

# A Simple, Powerful Method for Optimal Guidance of Spacecraft Formations

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## ABSTRACT

One of the most interesting and challenging aspects of formation guidance law design is the coupling of the orbit design and the science return. The analyst's role is more complicated than simply to design the formation geometry and evolution. He or she is also involved in designing a significant portion of the science instrument itself. The effectiveness of the formation as a science instrument is intimately coupled with the relative geometry and evolution of the collection of spacecraft. Therefore, the science return can be maximized by optimizing the orbit design according to a performance metric relevant to the science mission goals. In this work, we present a simple method for optimal formation guidance that is applicable to missions whose performance metric, requirements, and constraints can be cast as functions that are explicitly dependent upon the orbit states and spacecraft relative positions and velocities. We present a general form for the cost and constraint functions, and derive their semi-analytic gradients with respect to the formation initial conditions. The gradients are broken down into two types. The first type are gradients of the mission specific performance metric with respect to formation geometry. The second type are derivatives of the formation geometry with respect to the orbit initial conditions. The fact that these two types of derivatives appear separately allows us to derive and implement a general framework that requires minimal modification to be applied to different missions or mission phases. To illustrate the applicability of the approach, we conclude with applications to two missions: the Magnetospheric Multiscale mission (MMS), and the Laser Interferometer Space Antenna (LISA).

## INTRODUCTION

Developing guidance law algorithms for spacecraft formations and constellations is a challenging problem that can seemingly involve mutually exclusive goals. From the perspective of a mission analyst working on orbit design for distributed spacecraft missions, it is desirable to have a flexible method that can be applied to many problems, and that permits modification of the cost and constraints with as little effort as possible. We also desire an approach that can handle real-world mission constraints, find an optimal solution in a reasonable amount of time, and converge consistently with relatively poor initial guesses. The work presented in this paper was motivated by the need for an algorithm that can provide formation initial conditions for many different phases in the mission design process. These phases include, but are not limited to: developing initial conditions to use as target states to achieve after separation from the launch vehicle and separation from the spacecraft stack, developing new conditions for formation maintenance after perturbations and errors in navigation and control have caused a significant degradation in mission return, and developing new formation initial conditions for a new mission phase when the desired formation geometry is significantly different. For each of these mission phases we need to determine initial conditions for the orbits of multiple, cooperating spacecraft, that optimally satisfy mission requirements and constraints. This is a long list of goals, but we can come very close to meeting them all by developing effective design algorithms.

The approach presented in this paper is a direct parameter optimization method which assumes that at a given instant in time, one can formulate a performance metric that is an explicit function of the inertial cartesian states, and the relative position vectors, relative velocity vectors, range and range rates of all spacecraft in formation. The performance metric allows us to quantify the effectiveness of the relative dynamics of the spacecraft at a particular point in the orbit. The cost function,  $J$ , is simply the integral of the instantaneous performance metric over regions of interest to the particular mission. The goal is to maximize or minimize the cost function, according to the specific mission, while simultaneously satisfying equality and inequality constraints consistent with mission requirements and real-world limitations.

We begin by posing a continuous-time set of cost and constraint functions that define a general framework for the

problem. The continuous-time system is the problem we would like to solve. However, there are certain advantages to discretizing the problem. Hence, we discretize the problem to enable a Nonlinear Programming (NLP) routine to solve for an optimal solution. The discrete cost function can be calculated by quadrature, or a simple trapezoid rule, using numerical integration of the non-linear orbit equations of motion, with perturbations, to give us the state of the formation at discrete times in the orbit. The constraints are calculated in a similar manner. We show how to calculate the derivative of the cost function and the Jacobian of the constraint functions using the State Transition Matrix (STM). The STM is found by numerically integrating an additional 36 equations per spacecraft with terms that include perturbations. There is a significant advantage to this approach. Numerical integration, while non-trivial, is a solved problem and something that modern computers perform efficiently. Secondly, as we will see in a later section, the derivatives of the cost and constraints contain two types of terms. The first type of terms are composed of portions of the STM and inertial states. The second type of terms contain derivatives of algebraic functions with respect to the formation geometry. Because the terms are separated in this way, we can implement the algorithm in a very general way so as to allow minimal modification to new problems. We conclude the paper with two applications that illustrate how the method is applied to real-world problems. The first example is the Magnetospheric Multi-Scale (MMS) mission. The second example is the Laser Interferometer Space Antenna (LISA). Let's begin by defining some notation.

## SYMBOLS

### Variables

$\mathbf{r}$	Position vector
$\mathbf{v}$	Velocity vector
$n$	Number of spacecraft
$m$	Number of unique sides in the formation
$N$	Number of path constraints
$n_k$	Number of quadrature points
$c_k$	Quadrature constant at point $k$
$\mathbf{s}_{ik}$	Vector defining side $i$ at quadrature point $k$
$\dot{\mathbf{s}}_{ik}$	Rate vector of side $i$ , at point $k$
$s_{ik}$	Length of side $i$ , at point $k$
$\dot{s}_{ik}$	Rate of change side $i$ , at point $k$
$\Phi$	Orbit state transition matrix
$\mathbf{A}$	Upper left 3x3 partition of $\Phi$
$\mathbf{B}$	Upper right 3x3 partition of $\Phi$
$\mathbf{C}$	Lower left 3x3 partition of $\Phi$
$\mathbf{D}$	Lower right 3x3 partition of $\Phi$

### Subscripts

$i$	Side index
$k$	Quadrature point index
$j$	Spacecraft index
$\ell$	Spacecraft index
$p$	Dummy index
$J$	Indicates association with cost function
$C$	Indicates association with constraints

## General Problem Statement

The reliability and effectiveness of an optimization algorithm is intimately dependent upon the parameterization of the cost and constraint functions. In this section, we pose a set of cost and constraint functions for a formation of spacecraft as explicit functions of the cartesian states of the spacecraft in formation, and the relative motion of the spacecraft. Let's begin with some definitions. Recall that our goal is to find a set of initial cartesian states for a set

of  $n$  spacecraft. The vector defining the initial cartesian states for all spacecraft is

$$\mathbf{X}_o = [\mathbf{x}_{o1}^T \ \mathbf{x}_{o2}^T \ \dots \ \mathbf{x}_{on}^T]^T \quad (1)$$

where

$$\mathbf{x}_{o1} = [\mathbf{r}_{o1}^T \ \mathbf{v}_{o1}^T]^T = [x_{o1} \ y_{o1} \ z_{o1} \ \dot{x}_{o1} \ \dot{y}_{o1} \ \dot{z}_{o1}]^T \quad (2)$$

and so on. For our implementation, we choose to use the Earth MJ2000 equatorial coordinate system to express the cartesian states. However, it is possible to work in other coordinate systems, including rotating systems, as long as you are consistent. We assume that the equations of motion for the spacecraft are explicit functions of the spacecraft state and time, or:

$$\dot{\mathbf{X}}(t) = \mathbf{f}(\mathbf{X}, t) \quad (3)$$

Let's define the  $i^{th}$  unique side vector of the formation,  $\mathbf{s}_i$ , as the vector from the  $\ell^{th}$  to the  $j^{th}$  spacecraft

$$\mathbf{s}_i(t) = \mathbf{r}_j(t) - \mathbf{r}_\ell(t) \quad (4)$$

The subscripts  $j$  and  $\ell$  can be chosen arbitrarily as long as for each value of  $i$  there is a given  $j, \ell$  pair, the pairs result in a unique side,  $1 \leq j \leq n$  and  $1 \leq \ell \leq n$ , and the chosen definition is used consistently throughout the implementation. Similarly to the side vector, the side rate vector  $\dot{\mathbf{s}}_i$ , the side length  $s_i$ , and the side rate  $\dot{s}_i$  are defined as

$$\dot{\mathbf{s}}_i(t) = \dot{\mathbf{r}}_j(t) - \dot{\mathbf{r}}_\ell(t) \quad (5)$$

$$s_i(t) = (\mathbf{s}_i(t)^T \mathbf{s}_i(t))^{1/2} \quad (6)$$

$$\dot{s}_i(t) = \frac{\mathbf{s}_i(t)^T}{s_i(t)} \dot{\mathbf{s}}_i(t) \quad (7)$$

Let's pose a continuous-time cost function of the form

$$J(\mathbf{X}_o) = \int_{t_o}^{t_f} F_J(\mathbf{X}(t), \mathbf{s}_i(t), \dot{\mathbf{s}}_i(t), s_i(t), \dot{s}_i(t), \mathbf{C}_J) dt \quad (8)$$

where  $\mathbf{C}_J$  is a set of constants. Note that we assume the cost function is an explicit function of the relative geometry of the spacecraft. In Eq. (8),  $F_J$  is the performance metric that contains information on how well the geometry satisfies mission goals at a particular instant.  $F_J$  is mission specific and must be formulated according to a given problem. In a later section we present two examples for illustrative purposes. Similarly, we can define nonlinear equality constraints  $\mathbf{G}_E$ , and nonlinear inequality constraints,  $\mathbf{G}_I$ , as

$$\mathbf{G}_E(\mathbf{X}_o) = \mathbf{F}_E(\mathbf{X}(t), \mathbf{s}_i(t), \dot{\mathbf{s}}_i(t), s_i(t), \dot{s}_i(t), \mathbf{c}_E) = \mathbf{C}_E \quad (9)$$

$$\mathbf{G}_I(\mathbf{X}_o) = \mathbf{F}_I(\mathbf{X}(t), \mathbf{s}_i(t), \dot{\mathbf{s}}_i(t), s_i(t), \dot{s}_i(t), \mathbf{c}_I) \leq \mathbf{C}_I \quad (10)$$

Finally, we also impose linear equality constraints, linear inequality constraints, and bound constraints, respectively as

$$\mathbf{A}_E \mathbf{X}_o = \mathbf{B}_E \quad (11)$$

$$\mathbf{A}_I \mathbf{X}_o \leq \mathbf{B}_I \quad (12)$$

$$\mathbf{L} \leq \mathbf{X}_o \leq \mathbf{U} \quad (13)$$

We seek a solution that minimizes  $J$ , while simultaneously satisfying the constraints in Eqs. (9-13). However, there are several issues that make this difficult, if not impossible, to solve exactly. The first is that it is not possible to develop closed form solutions of  $\mathbf{X}$  in terms of  $t$ . Similarly, we don't know how to express the relative geometry in terms of explicit functions of time. Certain nonlinear constraint functions can also cause difficulties. By discretizing the problem, we can avoid these complications. Yet, we must be satisfied with a solution that is not exactly optimal. However, the difference between the rigorously optimal solution and the near-optimal solution obtained from parameter optimization is often so small as to make the difference academic. Now let's take a look at how to discretize the continuous-time problem.

## Discretization of the Cost and Constraints

In general, we cannot solve the integral in Eq. (8). Also, it is often difficult to evaluate the constraint functions in Eqs. (9-13) along the entire trajectory. Furthermore, differentiating the equations with respect to  $\mathbf{X}_o$  is usually difficult. For these reasons, it is convenient to approximate the integral in the cost function by using a quadrature rule. Similarly, it is convenient to evaluate path constraints at discrete points along the trajectory. Once we have recast the cost and constraints as a discrete system, we can take the derivatives relatively easily, as we'll see near the end of this section.

Let's begin by writing the cost function as a quadrature rule as follows:

$$J(\mathbf{X}_o) \approx C_J \sum_{k=1}^{n_k} c_k F_J(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \quad (14)$$

where  $k$  is the quadrature point index,  $n_k$  is the number of quadrature points,  $c_k$  is the quadrature constant associated with point  $k$ ,  $\mathbf{X}_k = \mathbf{X}(t_k)$ ,  $\mathbf{s}_{ik} = \mathbf{s}_i(t_k)$  and so on.

We have imposed many types of constraints in the continuous time system. Due to space limitations, we'll limit the discussion here to the most difficult of these constraints: nonlinear path constraints. Nonlinear point constraints, linear, and bound constraints are much easier to handle, and we leave them as an exercise for the reader. The nonlinear path constraints are constraints that must be satisfied along a section of the trajectory and not at a specific point. We can group the equality and inequality path constraints together for convenience as

$$\mathbf{G}(\mathbf{X}_o) = \begin{pmatrix} \mathbf{G}_E(\mathbf{X}_o) \\ \mathbf{G}_I(\mathbf{X}_o) \end{pmatrix} \approx \begin{pmatrix} P_{1k}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ P_{2k}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ \vdots \\ P_{\ell k}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ \vdots \\ P_{Nk}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \end{pmatrix} \quad (15)$$

where the function  $P_{\ell k}$  is the  $\ell^{th}$  path constraint function evaluated at the  $k^{th}$  quadrature point. Hence, we now have  $(N \cdot k)$  point constraints in the discrete time system, whereas we only had  $N$  path constraints in the continuous time system.

NLP algorithms are almost always more efficient when they are supplied with analytic derivatives as opposed to using finite differencing to find approximations for the derivatives. Recall that the parameters we allow the NLP algorithm to vary are the initial cartesian states,  $\mathbf{X}_o$ . We need to determine the derivative of the cost and constraints with respect to  $\mathbf{X}_o$ . We can write the derivative of the cost function as:

$$\frac{\partial J}{\partial \mathbf{X}_o} \approx C_J \sum_{k=1}^{n_k} c_k \frac{\partial}{\partial \mathbf{X}_o} F_J(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \quad (16)$$

Using the chain rule we can expand Eq.(16) to read

$$\frac{\partial J}{\partial \mathbf{X}_o} \approx C_J \sum_{k=1}^{n_k} c_k \left( \frac{\partial F_J}{\partial \mathbf{X}_k} \frac{\partial \mathbf{X}_k}{\partial \mathbf{X}_o} + \frac{\partial F_J}{\partial \mathbf{s}_{ik}} \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{X}_o} + \frac{\partial F_J}{\partial \dot{\mathbf{s}}_{ik}} \frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{X}_o} + \frac{\partial F_J}{\partial s_{ik}} \frac{\partial s_{ik}}{\partial \mathbf{X}_o} + \frac{\partial F_J}{\partial \dot{s}_{ik}} \frac{\partial \dot{s}_{ik}}{\partial \mathbf{X}_o} \right) \quad (17)$$

Similarly, for the constraints we can write

$$\frac{\partial \mathbf{G}}{\partial \mathbf{X}_o} = \begin{pmatrix} \frac{\partial P_{1k}}{\partial \mathbf{X}_o}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ \frac{\partial P_{2k}}{\partial \mathbf{X}_o}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ \vdots \\ \frac{\partial P_{\ell k}}{\partial \mathbf{X}_o}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \\ \vdots \\ \frac{\partial P_{Nk}}{\partial \mathbf{X}_o}(\mathbf{X}_k, \mathbf{s}_{ik}, \dot{\mathbf{s}}_{ik}, s_{ik}, \dot{s}_{ik}) \end{pmatrix} \quad (18)$$

Using the chain rule we can write the derivative of the  $\ell^{th}$  constraint at the  $k^{th}$  quadrature point as

$$\frac{\partial \mathbf{P}_{\ell k}}{\partial \mathbf{X}_o} \approx \frac{\partial \mathbf{P}_{\ell k}}{\partial \mathbf{X}_k} \frac{\partial \mathbf{X}_k}{\partial \mathbf{X}_o} + \frac{\partial \mathbf{P}_{\ell k}}{\partial \mathbf{s}_{ik}} \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{X}_o} + \frac{\partial \mathbf{P}_{\ell k}}{\partial \dot{\mathbf{s}}_{ik}} \frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{X}_o} + \frac{\partial \mathbf{P}_{\ell k}}{\partial s_{ik}} \frac{\partial s_{ik}}{\partial \mathbf{X}_o} + \frac{\partial \mathbf{P}_{\ell k}}{\partial \dot{s}_{ik}} \frac{\partial \dot{s}_{ik}}{\partial \mathbf{X}_o} \quad (19)$$

Inspecting Eqs. (17) and (19) we see that two types of terms appear and that these terms appear as pairs in a product. The first type of term is the derivative of the user supplied function with respect to the formation geometry. Examples of this type of constraint are

$$\frac{\partial F_J}{\partial \mathbf{X}_k}, \quad \frac{\partial F_J}{\partial \mathbf{s}_{ik}}, \quad \frac{\partial \mathbf{P}_{\ell k}}{\partial \mathbf{s}_{ik}} \quad (20)$$

The second type of term is the derivative of the geometry at a specific quadrature point, with respect to the formation initial conditions. Examples of this type of term are

$$\frac{\partial \mathbf{X}_k}{\partial \mathbf{X}_o}, \quad \frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{X}_o}, \quad \frac{\partial s_{ik}}{\partial \mathbf{X}_o} \quad (21)$$

What makes this method “simple” is two-fold. First, the terms of type one contain derivatives of the mission performance metric function with respect to the formation geometry. For many missions, the performance metric can be cast as an algebraic function of the formation geometry. As we’ll see in the examples presented in later sections, these derivatives are trivial. The second reason this method is “simple”, is that the second type of terms contain derivatives of the geometry with respect to the initial conditions, and the form of these derivatives is independent of the mission specific performance metric, and therefore they only have to be derived and implemented once! It is very convenient that the derivatives are separated in this way, and the fact that they are is simply due to the application of the chain rule to a cost function that is explicitly dependent upon the formation geometry. Let’s look at the derivatives of type two, which are the derivatives of the formation geometry and its evolution, with respect to the formation initial conditions.

## Derivatives of Formation Relative Geometry with Respect to Orbit Initial Conditions

The derivatives of the formation geometry with respect to the formation initial conditions contain information on the sensitivity of the state of the formation at some time,  $t_k$ , with respect to the state of the formation at the initial epoch. We would expect portions of the STM to appear in these derivatives. Recall that the definitions for the geometry given in Eqs. (4-7) express the relative geometry, such as the relative position vector between two spacecraft, in terms of the cartesian states of the spacecraft. To calculate the derivative of side vector  $\mathbf{s}_{ik}$ , with respect to the initial position of the  $p^{th}$  spacecraft,  $\mathbf{r}_{op}$ , we can write

$$\frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{r}_{op}} = \frac{\partial}{\partial \mathbf{r}_{op}} (\mathbf{r}_{jk} - \mathbf{r}_{\ell k}) = \begin{cases} \mathbf{A}_j(t_k, t_o) & p = j \\ -\mathbf{A}_\ell(t_k, t_o) & p = \ell \\ \mathbf{0} & \text{otherwise} \end{cases} \quad (22)$$

where  $\mathbf{A}_j$  is the upper left 3x3 component of the STM for the  $j^{th}$  spacecraft and so on. We define the four 3x3 subcomponents of the STM as

$$\mathbf{\Phi}(t, t_o) = \begin{pmatrix} \mathbf{A}_{3 \times 3}(t, t_o) & \mathbf{B}_{3 \times 3}(t, t_o) \\ \mathbf{C}_{3 \times 3}(t, t_o) & \mathbf{D}_{3 \times 3}(t, t_o) \end{pmatrix} \quad (23)$$

The derivative of  $\mathbf{s}_{ik}$  with respect to  $\mathbf{v}_{op}$  can be written as

$$\frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{v}_{op}} = \frac{\partial}{\partial \mathbf{v}_{op}} (\mathbf{r}_{jk} - \mathbf{r}_{\ell k}) = \begin{cases} \mathbf{B}_j(t_k, t_o) & p = j \\ -\mathbf{B}_\ell(t_k, t_o) & p = \ell \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

The remaining derivatives of the formation relative geometry with respect to the initial conditions are:

$$\frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{r}_{op}} = \frac{\partial}{\partial \mathbf{r}_{op}} (\mathbf{v}_{jk} - \mathbf{v}_{\ell k}) = \begin{cases} \mathbf{C}_j(t_k, t_o) & p = j \\ -\mathbf{C}_\ell(t_k, t_o) & p = \ell \\ 0 & \text{otherwise} \end{cases} \quad (25)$$

$$\frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{v}_{op}} = \frac{\partial}{\partial \mathbf{v}_{op}} (\mathbf{v}_{jk} - \mathbf{v}_{\ell k}) = \begin{cases} \mathbf{D}_j(t_k, t_o) & p = j \\ -\mathbf{D}_\ell(t_k, t_o) & p = \ell \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

$$\frac{\partial s_{ik}}{\partial \mathbf{r}_{op}} = \frac{\partial}{\partial \mathbf{r}_{op}} (\mathbf{s}_{ik}^T \mathbf{s}_{ik})^{1/2} = \frac{\mathbf{s}_{ik}^T}{s_{ik}} \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{r}_{op}} \quad (27)$$

$$\frac{\partial s_{ik}}{\partial \mathbf{v}_{op}} = \frac{\partial}{\partial \mathbf{v}_{op}} (\mathbf{s}_{ik}^T \mathbf{s}_{ik})^{1/2} = \frac{\mathbf{s}_{ik}^T}{s_{ik}} \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{v}_{op}} \quad (28)$$

$$\frac{\partial \dot{s}_{ik}}{\partial \mathbf{r}_{op}} = \frac{1}{s_{ik}} \left( \dot{\mathbf{s}}_{ik}^T - \mathbf{s}_{ik}^T \dot{\mathbf{s}}_{ik} \frac{\mathbf{s}_{ik}^T}{s_{ik}^2} \right) \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{r}_{op}} + \frac{\mathbf{s}_{ik}^T}{s_{ik}} \frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{r}_{op}} \quad (29)$$

$$\frac{\partial \dot{s}_{ik}}{\partial \mathbf{v}_{op}} = \frac{1}{s_{ik}} \left( \dot{\mathbf{s}}_{ik}^T - \mathbf{s}_{ik}^T \dot{\mathbf{s}}_{ik} \frac{\mathbf{s}_{ik}^T}{s_{ik}^2} \right) \frac{\partial \mathbf{s}_{ik}}{\partial \mathbf{v}_{op}} + \frac{\mathbf{s}_{ik}^T}{s_{ik}} \frac{\partial \dot{\mathbf{s}}_{ik}}{\partial \mathbf{v}_{op}} \quad (30)$$

The derivatives contained in Eqs. (22-30) only contain terms that involve the spacecraft states and their associated STMs. Therefore, these equations do not change from one problem to the next. What will change is the actual numbers in the STMs, and the spacecraft states. This enables us to implement the algorithm in a general way in order to minimize the amount of recoding to apply the algorithm to a new mission or mission phase. Now let's take a look at some aspects of the implementation we use. Then, we'll look at two mission applications.

## Implementation

The implementation of a numerical optimization approach must be carefully considered to ensure speed and ease of use. There are several aspects of the implementation that will have an influence on the speed of the approach we present in this paper. The first is how the orbit states and STMs are propagated. It is recommended that all propagation is performed in a compiled language as opposed to an interpreted language such as MATLAB. This is primarily due to the fact that we must propagate 42 differential equations per spacecraft. For our implementation, we coded most of the algorithm in MATLAB, and use MATLAB's *fmincon* SQP routine. However, all propagation is performed in compiled C code that can be called directly from MATLAB through a MEX interface.

The second issue that influences speed is the number of quadrature points chosen, and the considerable amount of bookkeeping which must be performed if the orbits are propagated and stored before evaluating the cost and constraints. For the examples presented in the next few sections, we used a simple trapezoid rule as opposed to a complex quadrature rule, in the summation of the discrete cost function. The number of points in the summation of the cost function was chosen to maximize the speed, while yielding an acceptable approximation to the exact, integral form of the cost function.

The second issue to consider is the organization of the code. To enable convenient application to new problems, it is helpful to isolate code that is specific to the user-defined function. By separating and organizing the code carefully, it is relatively simple to provide a new function containing cost and derivatives terms. For example, for the MMS example in the next section, there are only 16 lines of code that are specific to the MMS cost function, and they are all located in a single MATLAB function that is easy to modify to solve a new problem.

Now that we have discussed a few of the important issues in implementation, let's take a look at a specific mission example, the Magnetospheric MultiScale mission (MMS).

## Application: MMS

In the preceding portions of this paper we posed a general parameter optimization approach, and an approach to its implementation, to find optimal initial conditions of spacecraft formations. In this section, we'll discuss an application of the method to the MMS mission. We'll begin with a brief overview of the MMS mission. Then we'll present the cost and constraint functions we choose for MMS, and derive the necessary derivatives. We conclude the section with a discussion of results for a particular phase of the MMS mission.

The MMS mission is one of several missions in NASA's Solar Terrestrial Probes (STP) program. MMS will employ a four spacecraft formation to make fundamental advancements in our understanding of the Earth's magnetosphere and its dynamic interaction with the solar wind. MMS will not be the first mission to use multiple spacecraft to study magnetospheric dynamics. The Cluster II mission was successfully launched by the European Space Agency (ESA) in the summer of 2000 and has already provided fascinating results on magnetospheric dynamics. There are several phases in the MMS mission. In this work we focus on Phase I, which consists of a highly elliptic reference orbit with a perigee radius of  $1.2 R_e$  and an apogee radius of  $12 R_e$ . The mission design objective is to provide a near regular tetrahedron formation between true anomalies of  $160^\circ$  and  $200^\circ$ . There are many ways to pose a set of cost and constraint functions to achieve this goal and in the next section we'll discuss the method we choose. There have been many important contributions to the literature on tetrahedron formations and magnetospheric missions, and a detailed discussion of all the past efforts is beyond the scope of this work. References [1-7,9-14] contain information on the science of the MMS mission, reference orbit design, launch window analysis, tetrahedron design, and maneuver design among other topics.

### MMS Cost and Constraint Functions

In Phase I of the MMS mission, one of the mission design objectives is to provide a near regular tetrahedron, with sides of 10 km, within a region defined by  $\pm 20^\circ$  in true anomaly about orbit apogee. This requires casting a set of cost and constraint functions that take into account the size of the tetrahedron, and its likeness to a regular tetrahedron. Let's begin by investigating some useful properties of tetrahedrons. If we assume one of the spacecraft is the reference, then there are three relative position vectors that describe the relative geometry of the spacecraft and we'll define them as  $s_{1k}$ ,  $s_{2k}$ , and  $s_{3k}$  to be consistent with our previous definitions. Knowing these quantities, we can calculate the volume of the tetrahedron using

$$V_a = \frac{1}{6} |\mathbf{s}_{1k} \cdot (\mathbf{s}_{2k} \times \mathbf{s}_{3k})| \quad (31)$$

where  $V_a$  stands for the volume of the actual tetrahedron formed by the spacecraft, as opposed to the desired volume which we discuss below. There are six unique sides in the formation, and the side lengths can be used to calculate the average side length,  $L^*$ , using

$$L^* = \frac{1}{6} (s_{1k} + s_{2k} + s_{3k} + s_{4k} + s_{5k} + s_{6k}) \quad (32)$$

at time  $t_k$ . Paschmann<sup>10</sup> presents several methods for evaluating how close a tetrahedron is to being a regular tetrahedron. One approach is to compare the volume of the actual tetrahedron  $V_a$ , with the volume of a regular tetrahedron,  $V_r$ , with side lengths equal to the average side length of the actual tetrahedron. The volume of a regular tetrahedron of side length  $L$  can be calculated using

$$V_r = \frac{\sqrt{2}}{12} L^3 \quad (33)$$

Using  $V_a$  and  $V_r$ , Paschmann suggests an instantaneous volumetric performance metric,  $Q_v$ , for a tetrahedron of spacecraft as

$$Q_v^k = \frac{V_a^k}{V_r^k} \quad (34)$$

where the superscripts " $k$ " indicate values at the  $k^{th}$  point.  $Q_v$  has the useful property:  $0 \leq Q_v \leq 1$ . However, it does not take into account the actual size of the tetrahedron. We propose a simple polynomial function,  $Q_s$ , that has the properties below, to take into account the size of a tetrahedron. The constants  $\ell_1$ ,  $\ell_2$ ,  $\ell_3$ , and  $\ell_4$  are used to

change the shape of the function. A plot of  $Q_s$  for a 10 km tetrahedron, with  $\ell_1 = 4$ ,  $\ell_2 = 6$ ,  $\ell_3 = 18$ , and  $\ell_4 = 25$ , is shown in Fig. 1. The important property of  $Q_s$ , is that it is zero for tetrahedrons with an average side length of less than 4 km or greater than 25 km. Between the values of 6 km and 18 km,  $Q_s$  is equal to one.

$$Q_s^k(L^*) = \begin{cases} 0 & L^* < \ell_1 \\ (L^* - \ell_1)^2(L^* + \ell_1 - 2\ell_2)^2/(\ell_2 - \ell_1)^4 & \ell_1 < L^* < \ell_2 \\ 1 & \ell_2 < L^* < \ell_3 \\ (L^* - \ell_4)^2(L^* - 2\ell_3 + \ell_4)^2/(\ell_4 - \ell_3)^4 & \ell_3 < L^* < \ell_4 \\ 0 & L^* > \ell_4 \end{cases} \quad (35)$$

By making a composite function involving both  $Q_v$  and  $Q_s$  we can create a performance metric that evaluates both

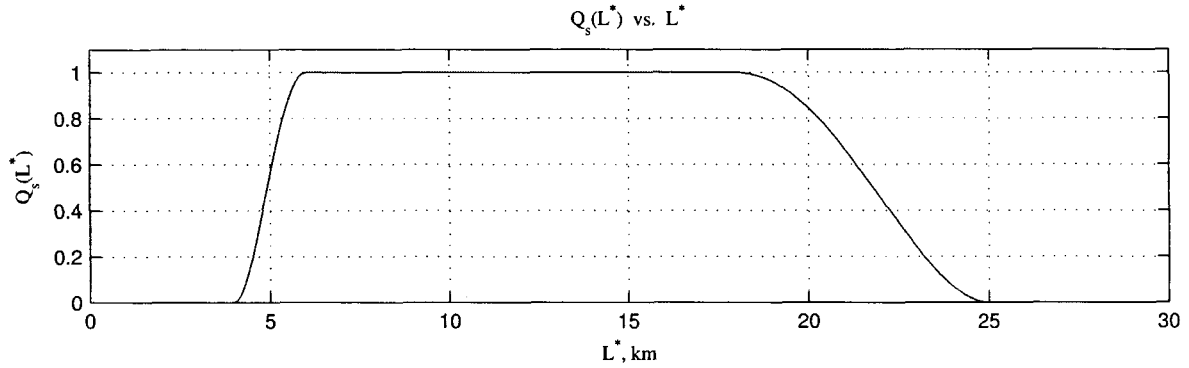


Figure 1: Plot of  $Q_s(L^*)$

the size and shape of a tetrahedron. Let's use the following continuous-time form

$$P(t) = 1 - Q_v(t)Q_s(t) \quad (36)$$

$P(t)$  will be zero for a regular tetrahedron with a side length between 6 and 18 km. For unacceptable tetrahedrons,  $P(t)$  is one. We can formulate a cost function by integrating  $P(t)$  over the region of interest, or

$$J = \int_{t_o}^{t_f} P(t)dt = \int_{t_o}^{t_f} (1 - Q_v(t)Q_s(t))dt$$

where  $t_o$  is the time when the reference spacecraft is at a true anomaly of  $160^\circ$ , and  $t_f$  is the time when the reference spacecraft is at a true anomaly of  $200^\circ$ . However, we can't evaluate this integral exactly, so we have to approximate it. We use a simple trapezoid rule, as opposed to a complicated quadrature rule, where

$$J \approx C \sum_{k=1}^{n_k} (1 - Q_v^k Q_s^k) \Delta t_k = C \sum_{k=1}^{n_k} \left( 1 - \frac{2}{\sqrt{2}} \frac{|\mathbf{s}_{1k} \cdot (\mathbf{s}_{2k} \times \mathbf{s}_{3k})|}{\sum_{i=1}^6 s_{ik}} \right) Q_{sk} \Delta t_k \quad (37)$$

For MMS, the independent variables are the initial cartesian states of the four spacecraft in formation. In addition to minimizing the cost function above, we also must satisfy the nonlinear constraints that the semimajor axes of all spacecraft in formation are equal. We can write this as

$$a_p = \frac{1}{\frac{2}{r_{op}} - \frac{v_{op}^2}{\mu}} = a_d \quad (38)$$

where  $a_p$  is the semimajor axis of the  $p^{th}$  spacecraft,  $a_d$  is the desired semimajor axis of all spacecraft,  $r_{op}$  is the magnitude of the initial position vector of the  $p^{th}$  spacecraft, and  $v_{op}$  is the magnitude of the initial velocity vector of the  $p^{th}$  spacecraft.

In Eq. (37), we have a performance metric for MMS that is an explicit function of the relative geometry of the formation. In Eqs. (38), we have four nonlinear constraint functions in terms of the initial spacecraft states. Now all we require are the derivatives of the cost and constraint functions in order to apply the method to find an optimal solution.

## Derivatives of MMS Cost and Constraint Functions

The last step to perform before solving for an optimal solution for MMS, is to determine the derivatives of the cost and constraints with respect to  $\mathbf{X}_o$ ,  $\mathbf{s}_{ik}$ ,  $\dot{\mathbf{s}}_{ik}$ ,  $s_{ik}$ , and  $\dot{s}_{ik}$ . The  $k^{th}$  term in the summation in the cost function,  $J_k$ , has the form

$$J_k = (1 - Q_v^k(s_{ik})Q_s^k(s_{ik})) \quad (39)$$

Notice that the cost function does not depend on the side rates, only on the side vectors and side lengths. Therefore

$$\frac{\partial J_k}{\partial \dot{s}_{ij}} = 0 \quad (40)$$

and

$$\frac{\partial J_k}{\partial \dot{s}_{ij}} = 0 \quad (41)$$

The derivatives of the cost function with respect to  $\mathbf{s}_{ik}$  are

$$\frac{\partial J_k}{\partial \mathbf{s}_{1k}} = -c \frac{2}{\sqrt{2}L^3} \frac{\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}}{|\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}|} (\mathbf{s}_{2k}^x \mathbf{s}_{3k})^T Q_s^k \quad (42)$$

$$\frac{\partial J_k}{\partial \mathbf{s}_{2k}} = c \frac{2}{\sqrt{2}L^3} \frac{\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}}{|\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}|} (\mathbf{s}_{1k}^T \mathbf{s}_{3k}^x) Q_s^k \quad (43)$$

$$\frac{\partial J_k}{\partial \mathbf{s}_{3k}} = -c \frac{2}{\sqrt{2}L^3} \frac{\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}}{|\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}|} (\mathbf{s}_{1k}^T \mathbf{s}_{2k}^x) Q_s^k \quad (44)$$

where the “x” superscript indicates a skew symmetric matrix.

Now let’s look at the derivatives of the cost function with respect to  $s_{ik}$ . We start with

$$\frac{\partial J_k}{\partial s_{ik}} = \frac{\partial}{\partial s_{ik}} (1 - Q_v^k Q_s^k) = -Q_v^k \frac{\partial Q_s^k}{\partial s_{ik}} - \frac{\partial Q_v^k}{\partial s_{ik}} Q_s^k \quad (45)$$

We can show that

$$\frac{\partial J_k}{\partial s_{ik}} = -\frac{Q_v^k}{6} \frac{\partial Q_s^k}{\partial L^*} + \frac{Q_s^k}{\sqrt{2}} \frac{|\mathbf{s}_{1k} \cdot \mathbf{s}_{2k} \times \mathbf{s}_{3k}|}{L^{*4}} \quad (46)$$

where

$$\frac{\partial Q_{sk}}{\partial L^*} = \begin{cases} 0 & L^* < \ell_1 \\ 4(L^* - \ell_1)(L^* + \ell_1 - 2\ell_2)(L^* - \ell_2)/(-\ell_2 + \ell_1)^4 & \ell_1 < L^* < \ell_2 \\ 0 & \ell_2 < L^* < \ell_3 \\ 4(L^* - 2\ell_3 + \ell_4)(L^* - \ell_4)(L^* - \ell_3)/(\ell_4 - \ell_3)^4 & \ell_3 < L^* < \ell_4 \\ 0 & L^* > \ell_4 \end{cases} \quad (47)$$

Finally, the nonzero derivatives of the semimajor axis constraints are

$$\frac{\partial a_p}{\partial \mathbf{r}_{op}} = \frac{\partial a_p}{\partial r_{op}} \frac{\partial r_{op}}{\partial \mathbf{r}_{op}} = 2 \left( \frac{a}{r_{op}} \right)^2 \frac{\mathbf{r}_{op}^T}{r_{op}} \quad (48)$$

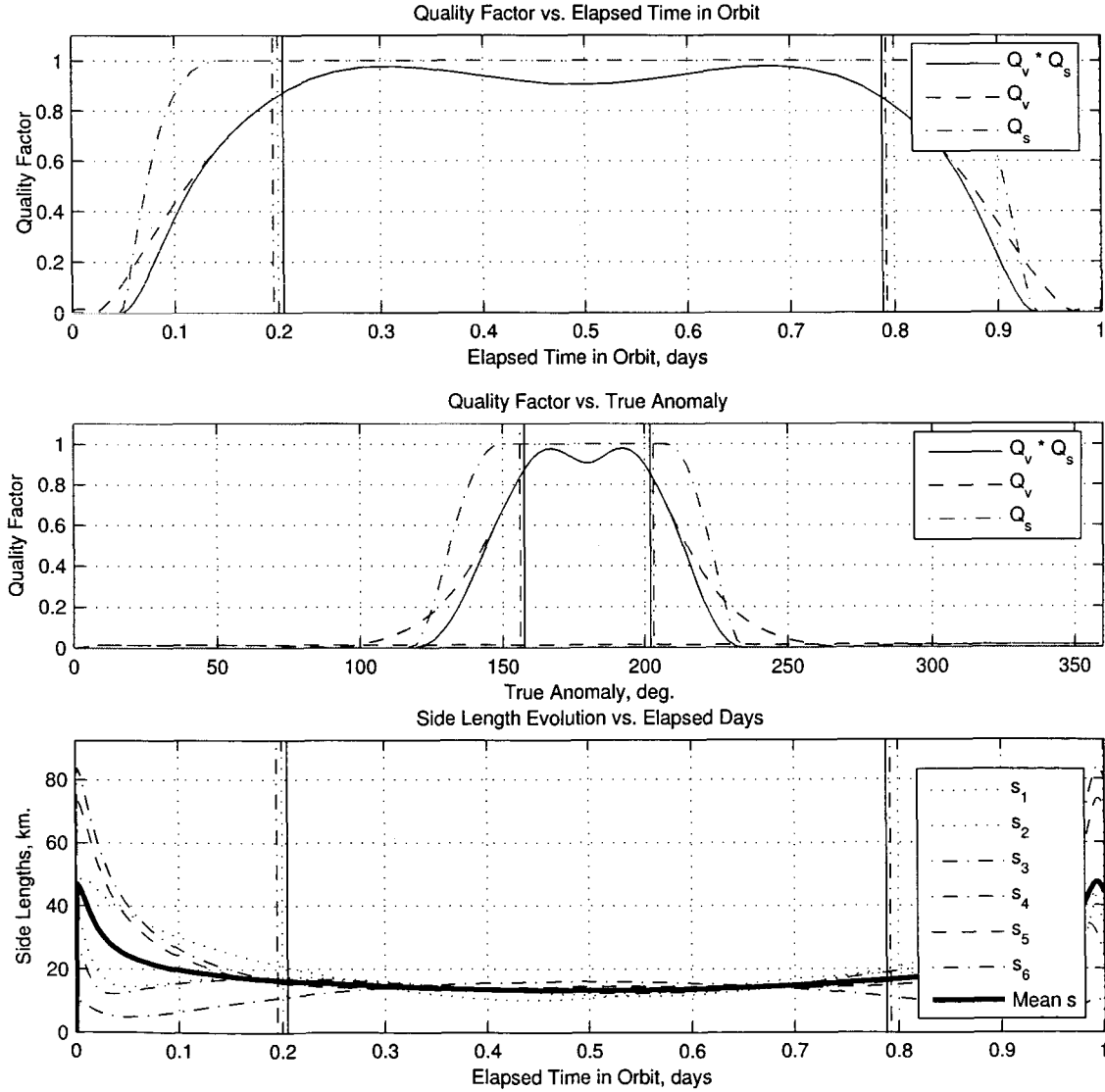
$$\frac{\partial a_p}{\partial \mathbf{v}_{op}} = \frac{\partial a_p}{\partial v_{op}} \frac{\partial v_{op}}{\partial \mathbf{v}_{op}} = \frac{2a_p^2}{\mu} \mathbf{v}_{op}^T \quad (49)$$

We now have all of the information to use an NLP code to find and optimal solutions. In the next section, we discuss the results for an optimal MMS formation.

## Results of Optimization for MMS

Recall that the primary formation flying goal for the phase of MMS we consider is to provide a near regular tetrahedron, with side lengths near 10 km, in the orbit region between  $160^\circ$  and  $200^\circ$  true anomaly. We also must satisfy the constraint that the semimajor axes for all spacecraft are the same. Figure 2 shows several plots that illustrate an optimal solution. The top figure shows the evolution of  $Q_s$ ,  $Q_v$ , and  $Q_s \cdot Q_v$  for one Keplerian orbit. The lower plot shows the evolution of the six individual sides and the average side length. We see that the quality factor is near one during the region of interest, which is bounded by the vertical dashed lines. The optimal solution is significantly better than the initial guess, which was calculated using an algorithm described in Ref. [9]. For the initial guess, the time averaged quality factor  $Q_v \cdot Q_s$  was about 0.55. The NLP algorithm found an optimal solution where the time-averaged quality factor was about 0.94. For this solution, the run time was about 35 seconds, and required about 40 function evaluations. From inspection of the bottom plot in Fig. 2 we see that the average side length is on the order of 10 km. The orbit states for this solution are found in the Appendix.

**Figure 2:** Results of MMS Orbit Optimization



Now let's take a look at a more complicated example, the Laser Interferometer Space Antenna (LISA), which has a large number of nonlinear path constraints.

## Application: LISA

LISA is a NASA/ESA mission to detect and study gravitational waves from massive black hole systems and galactic binaries. Gravitational waves are ripples in space-time caused by massive objects. They are a prediction of general relativity, but are yet to be directly detected. The detection and understanding of gravitational waves can provide breakthroughs and refinements in current relativistic theory. For a more detailed discussion of the LISA mission, see Refs. [8, 15-17].

The nominal LISA formation consists of three spacecraft in heliocentric orbits trailing the Earth by about  $20^\circ$ , with inclinations near  $1^\circ$  with respect to the ecliptic plane. The mission design goals for LISA are challenging. The primary goal is to provide a formation that maintains a nearly equilateral triangle with sides near  $5 \times 10^6$  km for the entire life of the mission, which is currently about 8.5 years. This has to be achieved entirely through careful orbit design, as continuous feedback control of the orbits is not permitted because it will interfere with the science measurements. We also must ensure that the sides of the triangle remain within one percent of  $5 \times 10^6$  km, and that the side rates never exceed 15 m/s. There is a secondary, and competing goal, that we keep the formation as close to Earth as possible for power reasons. Let's look at how we can cast these goals and constraints in a manner consistent with the algorithm presented in previous sections.

### LISA Cost and Constraint Functions

There are essentially two goals for LISA that can be formulated as a cost function. The first is the desire to keep the side lengths as close to  $5 \times 10^6$  km as possible. The second is to minimize the distance between the LISA formation and the Earth. We choose to pose the cost function in terms of the Earth distance, and cast the side length goals as constraints. The following cost function permits us to minimize the distance of the LISA formation from Earth:

$$J = c \sum_{k=1}^{n_k} (r_{1k} + r_{2k} + r_{3k}) \quad (50)$$

where  $r_{1k} = (x_{1k}^2 + y_{1k}^2 + z_{1k}^2)^{1/2}$  and so on. To ensure that the sides of the formation never exceed a one percent variation from the desired side length, we can apply the following set of  $i \cdot k$  path constraints.

$$(s_{ik} - L)^2 \leq g_1(t_k)^2 \quad (51)$$

where  $L = 5 \times 10^6$  and

$$g_1(t_k) = \begin{cases} -2000t_k + 50,000 \text{ km} & t_k < 5 \text{ yrs} \\ 40,000 \text{ km} & t_k \geq 5 \text{ yrs} \end{cases} \quad (52)$$

where  $t_k$  is the time at the  $k^{th}$  point, in units of years. The function  $g_1(t_k)$  above, and  $g_2(t_k)$  below, were suggested by Sweetser.<sup>17</sup>  $g_1(t_k)$  and  $g_2(t_k)$  are chosen so that the range and range rate requirements are still satisfied with expected orbit insertion errors. The following set of  $i \cdot k$  constraints ensures that the side rates never exceed 15 m/s:

$$(\dot{s}_{ik})^2 \leq g_2(t_k)^2 \quad (53)$$

where

$$g_2(t) = \begin{cases} -\frac{3}{5}t_k + 15 \text{ m/s} & t_k < 5 \text{ yrs} \\ 12 \text{ m/s} & t_k \geq 5 \text{ yrs} \end{cases} \quad (54)$$

Finally, we require that the ecliptic inclination,  $i_{ec}$ , of each orbit is less than one degree.

$$i_{ec} < 1^\circ \quad (55)$$

Let's reformulate this constraint so that the derivatives are simpler.

$$\cos i_{ec} = \hat{\mathbf{z}}^T \hat{\mathbf{h}}_{ec} > \cos 1^\circ \quad (56)$$

where

$$\hat{\mathbf{z}} = [0 \ 0 \ 1]^T \quad (57)$$

$$\mathbf{h}_{ec} = \mathbf{r}_{ec} \times \mathbf{v}_{ec} \quad (58)$$

and  $\mathbf{r}_{ec}$  and  $\mathbf{v}_{ec}$  are the initial position and velocity of a LISA spacecraft, in the Sun-centered mean ecliptic of J2000 frame, and are given by

$$\mathbf{r}_{ec} = \mathbf{r}_{E/S} + \mathbf{R}\mathbf{r}_{ok}; \quad (59)$$

$$\mathbf{v}_{ec} = \mathbf{v}_{E/S} + \mathbf{R}\mathbf{v}_{ok}; \quad (60)$$

where  $\mathbf{r}_{E/S}$  and  $\mathbf{v}_{E/S}$  are the position and velocity of the Earth with respect to the Sun, at the initial epoch, and  $\mathbf{R}$  is the rotation matrix from the Earth's mean equator of J2000 to the mean ecliptic of J2000.

The derivatives of the cost and constraint functions are presented, without derivation, in Table 1. The leftmost column in the table indicates the function, and the rows are the derivatives with respect to the term that appears in the column labels. The superscript "x" indicates the skew symmetric matrix. For derivatives with respect to the initial cartesian states, the "p" subscript is a dummy variable associated with the state of the  $p^{th}$  spacecraft.

**Table 1:** Derivatives of LISA Cost and Constraint Functions

Function	$\partial s_{ik}$	$\partial \dot{s}_{ik}$	$\partial \mathbf{r}_{op}$	$\partial \mathbf{v}_{op}$
$\partial c \sum_{k=1}^{n_k} (r_{1k} + r_{2k} + r_{3k})$	0	0	$\frac{\mathbf{r}_{pk}^T}{r_{pk}} \mathbf{A}_p(t_k, t_o)$	$\frac{\mathbf{r}_{pk}^T}{r_{pk}} \mathbf{B}_p(t_k, t_o)$
$\partial (s_{ik} - L)^2$	$2(s_{ik} - L)$	0	0	0
$\partial (\dot{s}_{ik})^2$	0	$2\dot{s}_{ik}$	0	0
$\partial (\hat{\mathbf{x}}^T) \mathbf{h}_{ec}$	0	0	$\mathbf{z}^T (-\mathbf{v}_{E/S}^x \mathbf{R} - \mathbf{R} \mathbf{v}_{op}^x)$	$\mathbf{z}^T (\mathbf{r}_{E/S}^x \mathbf{R} + \mathbf{R} \mathbf{r}_{op}^x)$

Let's discuss some practical considerations before moving on to the results of the LISA optimization. The current launch date for LISA is Jan. 2015. The initial epoch we use is 01 Jan. 2015 12:00:00.000 UTC. The force model used in the LISA optimization includes point masses from all planets and the Earth's moon. The coordinate system is Earth's mean J2000 equator. However, the converged solution is presented in the Appendix in the sun-centered mean ecliptic of J2000. Now let's look at details of a representative initial guess and optimal solution for LISA.

## Results of Optimization for LISA

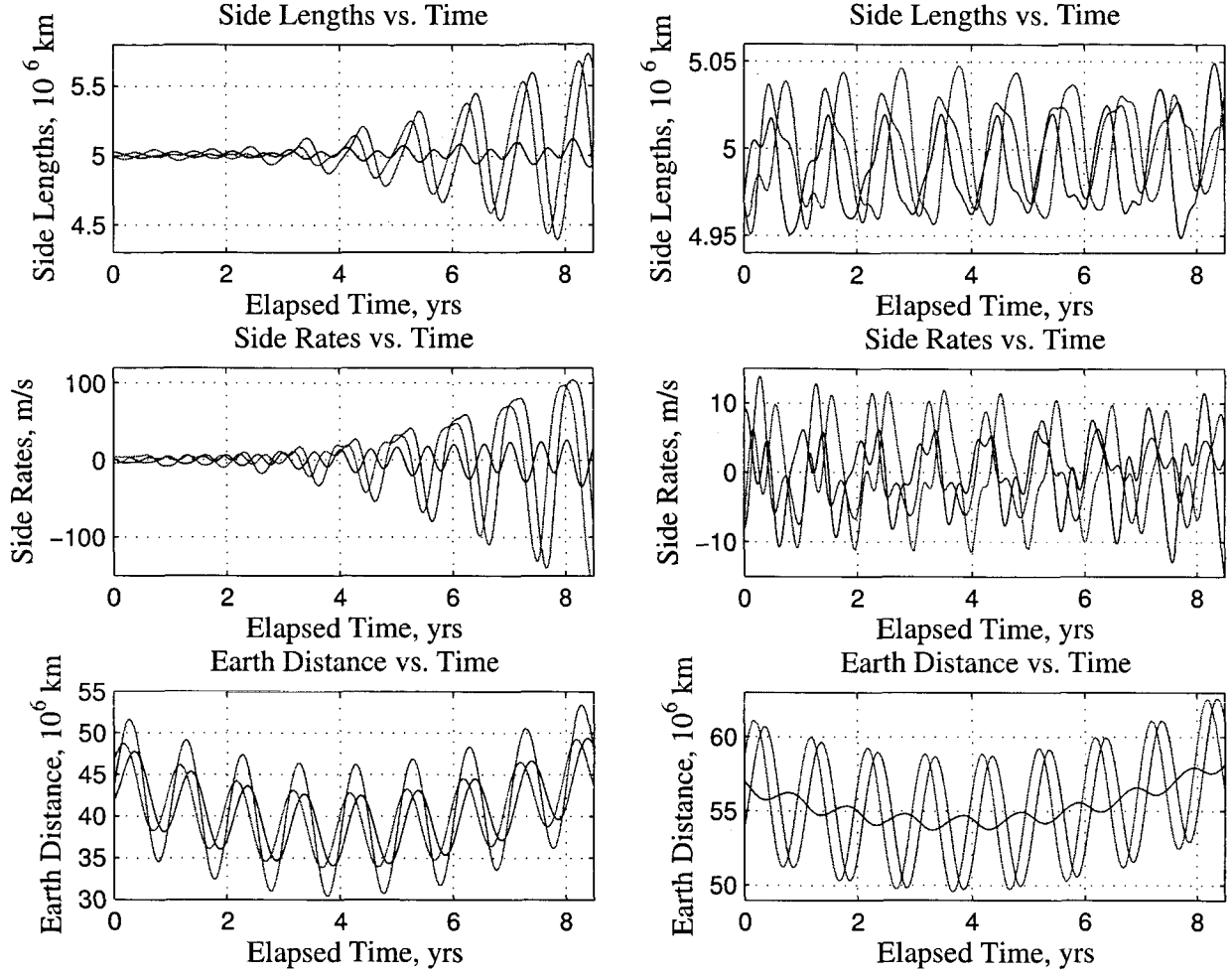
Figure 3 contains plots illustrating characteristics of the initial guess and converged solution for an 8.5 year LISA trajectory. The plots associated with the initial guess are in the left column, the plots associated with the converged solution are in the right column. The independent variable for each plot is the elapsed mission time in years. The first row of plots shows the side lengths between spacecraft, or  $s_i$ . The second row of plots shows the side rates, or  $\dot{s}_i$ . Finally, the third row contains plots illustrating the evolution of the spacecraft-to-Earth distance.

Let's begin by taking a closer look at some properties of the initial guess. The first point we see is that the initial guess is not a feasible solution. We see that the maximum variation in range between any two LISA spacecraft is around 700,000 km. This is over an order of magnitude larger than the desired maximum range variation of 50,000 km. Similarly, the maximum range rate for the initial guess is around 140 m/s, which is about an order of magnitude higher than the 15 m/s constraint. From inspection of the plots, we see that the initial guess is unstable. It turns out that the instability is due to the fact that the initial guess has been placed too close to the Earth. We see that the distance from Earth to the formation varies from a minimum of about  $31 \times 10^6$  km, to about  $53 \times 10^6$  km.

The optimal solution is found to have significantly improved properties over the initial guess. Table 2 contains a comparison of relevant statistics for the two cases. We see that the maximum variation in range between any two spacecraft is 50,000 km for the converged solution. This indicates that the range constraint is active at the solution. The range rate between any two spacecraft is 13.7 m/s. The penalty we have paid to satisfy these constraints, is that the formation was moved farther away from Earth. The minimum and maximum distance from any LISA spacecraft to the Earth, is about  $49 \times 10^6$  km and  $64 \times 10^6$  km respectively.

In summary, we have seen that the method we present here can solve a complex mission design problem, with multiple real-world constraints, in a deep-space flight regime. We began with a set of goals and requirements for

**Figure 3:** Initial Guess and Converged Solution for LISA Optimization



the LISA mission, and cast them mathematically as a set of cost and constraint functions. Next, we presented the derivatives of the cost and constraint functions with respect to the formation geometry. Comparing the initial guess to the converged solution, indicates that we can satisfy the current mission requirements for LISA, and that we don't need a feasible guess to find a solution. Now let's look at some general conclusions we can draw from this work.

**Table 2:** Statistics for LISA Initial Guess and Converged Solution

Trajectory	$\max( \dot{s}_{ik} )$ (m/s)	$\max(s_{ik})$ ( $10^6$ km)	$\min(s_{ik})$ ( $10^6$ km)	$\max(r_{ik})$ ( $10^6$ km)
Initial Guess	140	5.7	4.4	53
Converged Solution	13.7	5.04	4.95	64

## Conclusions

We began this work with a long list of objectives that is motivated by the need for a practical, efficient method to solve complex, real-world guidance problems to support a diverse set of formation flying missions. The approach we developed and presented meets many, if not all of our initial goals. The approach is non-linear and not restricted to specific flight regimes or small inter-spacecraft separations. By carefully implementing the method, we can minimize the amount of work that is required to solve new problems. For example, the entire MATLAB implementation of the cost and constraint functions for the MMS example, neglecting code for propagation and the NLP solver, consists of only 354 lines of code. Only 36 lines are specific to the MMS problem. The reason so few lines of code are specific to the mission problem is due to the fact that we assumed the cost function is an explicit function of the spacecraft relative positions, velocities, ranges, and range rates. This simple assumption allows us to perform half of the derivation of the derivatives a-priori in a general way, and implement them in software only once. This reduces the work that must be performed to solve new analysis problems. Furthermore, the derivatives that can be calculated a-priori are the derivatives of the dynamics with respect to the initial conditions. The derivatives that are problem dependent, are often simply derivatives of algebraic functions with respect to the formation geometry.

We illustrated the applicability of the approach to two different missions: MMS and LISA. Both of these missions have a complex set of cost and constraint functions that are explicitly dependent upon the formation geometry. We presented optimal orbit solutions for both missions.

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## Appendix: Optimal Orbit States for MMS and LISA

**Table 3:** MMS Results: State Information for a Representative, Phase I, 10 km Tetrahedron Formation

Property	MMS1	MMS2	MMS3	MMS4
$a$ (km)	42095.7	42095.7	42095.7	42095.7
$e$	0.81818	0.81798411	0.81799342	0.81800317
$i$ (deg.)	27.8	27.800078	27.801283	27.798145
$\omega$ (deg.)	15.000001	15.008968	15.002286	14.982171
$\Omega$ (deg.)	0	359.99962	359.98943	0.014249285
$\nu$ (deg.)	180	179.99657	180.00303	180.00225

**Table 4:** LISA Results in Heliocentric Mean Ecliptic J2000, 01 Jan 2015 12:00:00.000 UTC

OE	LISA1	LISA2	LISA3	LISA-Center
$a$ (km)	149457958	149457278	149457372	149457536
$e$	0.018265676	0.00949038651	0.00887263317	0.0086422348
$i$ (deg.)	0.890058699	0.993559828	0.993293398	0.0663052608
$\omega$ (deg.)	90.6009132	32.5150215	148.24379	272.489236
$\Omega$ (deg.)	3.64713792	126.757235	240.28666	182.897519
$\nu$ (deg.)	343.945562	278.418085	51.016237	-16.9098743